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Note on a paper by Robinet, Gressier, Casalis & Moschetta

By JEAN-FRANÇOIS COULOMBEL, SYLVIE BENZONI-GAVAGE AND DENIS SERRE

UMPA, CNRS – UMR 5669, École Normale Supérieure de Lyon, 46 allée d'Italie, 69364 Lyon CEDEX 07, France

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In Robinet *et al.* (2000), an instability for shock waves in gas dynamics is exhibited. The fluid was assumed to obey the polytropic perfect gas pressure law. We show in this note that such an instability does not occur, as proved in the extensive work of Majda (1983). Two arguments are developed: we give first a mathematical result that is violated by the conclusions of Robinet *et al.* (2000). Then we detail the results of Robinet *et al.* (2000) and show why they do not yield any conclusions. Finally, we develop a general calculation and show that an instability cannot occur.

1. A theoretical argument

Following the pioneering works of D'yakov (1954), Erpenbeck (1962) and Kontorovič (1958), Majda (1983) proved that planar shock waves are always uniformly stable when the fluid obeys a polytropic perfect gas pressure law. Details of the complete calculations of Majda (1983) may be found in Jenssen & Lyng (2002). We refer to Menikoff & Plohr (1989) for a review of the stability of shock waves in real gas dynamics. Note that Majda's result also holds for isentropic gas dynamics, see Majda (1983) or Majda (1984) for a review. The existence of 'branching points' (where the instability mentioned by Robinet *et al.* is supposed to occur) is pointed out and it is clearly shown that they do not give rise to any instability.

Via a normal modes analysis, the stability of a planar shock wave amounts to checking that a certain function Δ (called the Lopatinskii determinant), that depends on a wave vector k_y and a complex frequency ω, does not vanish when the imaginary part of ω is positive (whatever the wave vector is). Since the quantities involved depend holomorphically on the frequency ω, $\Delta(k_y, \cdot)$ can be defined as a holomorphic function on the half-plane $\{\mathrm{Im}\,\omega > 0\}$. Note that Δ also depends continuously on both states ahead and behind the shock front.

If there exists a non-zero wave vector k_y and a complex frequency ω of positive imaginary part such that $\Delta(k_y, \omega) = 0$, then the same property holds for close planar shocks (that is if we perturb the upstream and downstream states), see Benzoni-Gavage *et al.* (2002). With this statement in mind, it is clear that the result of Robinet *et al.* (2000) cannot hold because the exhibited instability meets the previous assumptions but only for isolated values of the Mach numbers.

2. The calculation of Robinet

We keep the notation of Robinet *et al.* and detail the computations of § 2.8: one has $\omega = \mathrm{i}k_y\overline{U}_1$ and

$$k_x^{(1)} = -\mathrm{i}k_y\frac{1+\overline{M}_1^2}{1-\overline{M}_1^2}, \quad k_x^{(2)} = k_x^{(3)} = k_x^{(4)} = \mathrm{i}k_y.$$

The subspace $\mathrm{Ker}(\boldsymbol{M}_1 - k_x^{(2)}\boldsymbol{M}_2)$ is spanned by the two vectors

$$\boldsymbol{V}_2 = \left(1, -\frac{\bar{\rho}_1}{\overline{T}_1}, 0, 0\right)^t \quad \text{and} \quad \boldsymbol{V}_3 = (0, 0, 1, -\mathrm{i})^t.$$

Note that

$$\boldsymbol{V}_4 = \left(\frac{\mathrm{i}\overline{U}_1}{rk_y}, 0, \frac{1}{k_y}, \frac{-(1+\mathrm{i})}{k_y}\right)^t$$

is a solution of the equation

$$(\boldsymbol{M}_1 - k_x^{(2)}\boldsymbol{M}_2)\boldsymbol{V}_4 = \boldsymbol{M}_2\boldsymbol{V}_3.$$

Thus the $\boldsymbol{V}_i$ form a Jordan basis of $\boldsymbol{M}_2^{-1}\boldsymbol{M}_1$ ($\boldsymbol{V}_1$ is an eigenvector for the eigenvalue $k_x^{(1)}$ but is of no use in the stability analysis). The problem reduces to determining whether there exist complex coefficients (C_2, C_3, C_4, X) such that

$$\left.\begin{aligned} \overline{U}_1\rho_f + \bar{\rho}_1 u_f - \overline{U}_1(\bar{\rho}_1 - \bar{\rho}_0)k_yX &= 0, \\ (r\overline{T}_1 + \overline{U}_1^2)\rho_f + r\bar{\rho}_1 T_f + 2\bar{\rho}_1\overline{U}_1 u_f &= 0, \\ v_f + \mathrm{i}(\overline{U}_1 - \overline{U}_0)k_yX &= 0, \\ C_pT_f + \overline{U}_1u_f - \overline{U}_1(\overline{U}_1 - \overline{U}_0)k_yX &= 0, \end{aligned}\right\} \tag{1}$$

where T_f, ρ_f, u_f and v_f are defined by

$$(T_f, \rho_f, u_f, v_f)^t = C_2\boldsymbol{V}_2 + C_3\boldsymbol{V}_3 + C_4\boldsymbol{V}_4. \tag{2}$$

In Robient *et al.* (2000, p. 248), the authors claim that a non-trivial solution exists when the upstream and downstream Mach numbers are given by

$$\overline{M}_0^2 = \frac{5+\gamma}{3-\gamma} \quad \text{and} \quad \overline{M}_1^2 = \tfrac{1}{3}.$$

C_2, C_3 and C_4 are given by

$$C_2 = \frac{\overline{U}_1(\overline{U}_1 - \overline{U}_0)}{\gamma r}\frac{2(2\gamma-1)}{(2\gamma-1)\overline{M}_1^2 + 3}k_yX,$$

$$C_3 = -(\overline{U}_1 - \overline{U}_0)\frac{2(1+\mathrm{i})}{(2\gamma-1)\overline{M}_1^2 + 3}k_yX,$$

$$C_4 = k_y(\overline{U}_1 - \overline{U}_0)\frac{2\mathrm{i}}{(2\gamma-1)\overline{M}_1^2 + 3}k_yX.$$

With these expressions, we obtain

$$v_f = -\mathrm{i}C_3 - \frac{1+\mathrm{i}}{k_y}C_4$$
$$= \mathrm{i}(\overline{U}_1 - \overline{U}_0)\frac{2(1+\mathrm{i})}{(2\gamma-1)\overline{M}_1^2+3}k_yX - (1+\mathrm{i})\frac{2\mathrm{i}(\overline{U}_1-\overline{U}_0)}{(2\gamma-1)\overline{M}_1^2+3}k_yX = 0.$$

From the linearized Rankine–Hugoniot conditions, we obtain

$$v_f = \mathrm{i}k_y(\overline{U}_1 - \overline{U}_0)X = 0$$

and thus $X = 0$. As a consequence, we obtain $C_2 = 0$, $C_3 = 0$ and $C_4 = 0$. Thus an instability associated with the complex frequency $\omega = \mathrm{i}k_y\overline{U}_1$ does not occur.

3. A general calculation

More generally, we shall show that under the assumption $\gamma \leqslant 2$, the only solution to equations (1)–(2) is the trivial solution: this result is independent of the Mach numbers (the only inequality we shall use is $\overline{M}_1 < 1$). Instead of the previous definition of $\boldsymbol{V}_4$, we set

$$\boldsymbol{V}_4 = \left(\frac{\overline{U}_1}{r}, 0, 1, 0\right)^t.$$

This amounts to adding $(-1+\mathrm{i})/k_y\boldsymbol{V}_3$ to the previous definition and multiplying by $(-\mathrm{i}k_y)$. With this new expression for $\boldsymbol{V}_4$, we obtain

$$T_f = C_2 + \frac{\overline{U}_1}{r}C_4, \qquad \rho_f = -\frac{\bar{\rho}_1}{\overline{T}_1}C_2, \qquad u_f = C_3 + C_4, \qquad v_f = -\mathrm{i}C_3.$$

Recall that C_p is given by

$$C_p = \frac{\gamma r}{\gamma - 1} = \frac{\bar{a}_1^2}{(\gamma-1)\overline{T}_1},$$

where a denotes the sound speed in the fluid. If we set $Y = k_yX$, we now need to solve the linear system

$$\left.\begin{aligned}
-\frac{\bar{\rho}_1\overline{U}_1}{\overline{T}_1}C_2 + \bar{\rho}_1C_3 + \bar{\rho}_1C_4 - \overline{U}_1(\bar{\rho}_1 - \bar{\rho}_0)Y &= 0, && (3)\\
-\frac{\overline{U}_1}{\overline{T}_1}C_2 + 2C_3 + 3C_4 &= 0, && (4)\\
C_3 - (\overline{U}_1 - \overline{U}_0)Y &= 0, && (5)\\
\frac{\bar{a}_1^2}{(\gamma-1)\overline{T}_1}C_2 + \overline{U}_1C_3 + \frac{2\gamma-1}{\gamma-1}\overline{U}_1C_4 - \overline{U}_1(\overline{U}_1 - \overline{U}_0)Y &= 0. && (6)
\end{aligned}\right\}$$

With the help of equation (5), equation (6) becomes

$$-\frac{\overline{U}_1}{\overline{T}_1}C_2 = (2\gamma-1)\overline{M}_1^2C_4.$$

We now place the previous expression for C_2 and the expression for C_3 (deduced from equation (5)) into equations (3) and (4). This yields the following linear system in (C_4, Y):

$$\left.\begin{aligned} \bar\rho_1[(2\gamma-1)\overline{M}_1^2+1]C_4+(\bar\rho_1+\bar\rho_0)(\overline{U}_1-\overline{U}_0)Y &= 0,\\ [(2\gamma-1)\overline{M}_1^2+3]C_4+2(\overline{U}_1-\overline{U}_0)Y &= 0. \end{aligned}\right\} \tag{7}$$

Here we have used the relation $\bar\rho_1\overline{U}_1=\bar\rho_0\overline{U}_0$ which is a consequence of the Rankine–Hugoniot jump conditions.

There exists a non-trivial solution (C_2, C_3, C_4, X) if and only if there exists a non-trivial solution of (7). This amounts to requiring that the determinant of this 2×2 system vanishes. This determinant is proportional to

$$[(2\gamma-1)\overline{M}_1^2-1]\bar\rho_1-[(2\gamma-1)\overline{M}_1^2+3]\bar\rho_0,$$

and we conclude that an instability occurs if and only if

$$[(2\gamma-1)\overline{M}_1^2-1]\bar\rho_1=[(2\gamma-1)\overline{M}_1^2+3]\bar\rho_0. \tag{8}$$

Equation (8) cannot hold if $(2\gamma-1)\overline{M}_1^2\leqslant 1$, so we assume $(2\gamma-1)\overline{M}_1^2>1$ and that (8) holds. We obtain

$$\bar\rho_1=\alpha\bar\rho_0 \quad \text{and} \quad \overline{U}_0=\alpha\overline{U}_1,$$

where α is a positive number given by

$$\alpha=\frac{(2\gamma-1)\overline{M}_1^2+3}{(2\gamma-1)\overline{M}_1^2-1}=1+\frac{4}{(2\gamma-1)\overline{M}_1^2-1}.$$

Since $\gamma\leqslant 2$ and $\overline{M}_1<1$, we obtain $\alpha>3$.

Recall that the planar shock wave satisfies the Rankine–Hugoniot conditions:

$$\begin{aligned} \bar\rho_1\overline{U}_1&=\bar\rho_0\overline{U}_0,\\ \bar\rho_0\overline{U}_0^2-\bar\rho_1\overline{U}_1^2&=r(\bar\rho_1\overline{T}_1-\bar\rho_0\overline{T}_0),\\ \frac{\gamma r}{\gamma-1}(\overline{T}_1-\overline{T}_0)&=\tfrac{1}{2}(\overline{U}_0^2-\overline{U}_1^2). \end{aligned}$$

From these equalities, we obtain the relation

$$\bar a_1^2=\gamma r\overline{T}_1=\gamma\alpha\overline{U}_1^2-\frac{\gamma-1}{2}(1+\alpha)\overline{U}_1^2,$$

and we have assumed that

$$\bar a_1^2<(2\gamma-1)\overline{U}_1^2.$$

We obtain therefore the inequality

$$\gamma\alpha-\frac{\gamma-1}{2}(1+\alpha)<2\gamma-1,$$

that is

$$\alpha<\frac{5\gamma-3}{\gamma+1}=5-\frac{8}{\gamma+1}<3.$$

This yields a contradiction, and therefore $C_4=0$ and $Y=0$. As a consequence $C_2=0$, $C_3=0$. The planar shock is stable, as stated in Majda (1983).

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